JOURNAL OF APPROXIMATION THEORY 65, 231-244 (1991)

Approximating Measures Invariant under Higher-Dimensional Chaotic Transformations

A. BOYARSKY* AND Y. S. LOU

Department of Mathematics, Concordia University, 7141 Sherbrooke St. W., Montreal, Quebec, Canada H4B 1R6

Communicated by Paul Nevai

Received January 24, 1990, revised August 3, 1990

Let τ be a Jablonski transformation from the n-dimensional unit cube into itself. We present a method for approximating the absolutely continuous invariant measures by means of approximating the Frobenius-Perron operator by finite-dimensional operators. This proves an n-dimensional version of a conjecture by Ulam and generalizes the one-dimensional results of T. Y. Li. © 1991 Academic Press, Inc.

1. INTRODUCTION

Let $I = [0, 1]$ and let $\tau: I^n \to I^n$ be a piecewise expanding transformation. For $n = 1$, Lasota and Yorke [12] proved the existence of an absolutely continuous invariant measure (acim) μ with respect to Lebesgue measure. Since then there have been a number of generalizations of this result to higher dimensions [3, 5, 10, 16]. If f is the density of μ with respect to Lebesgue measure m on I^n, then it is well known that f is the fixed point of the Frobenius–Perron operator P_τ. However, solving the resulting functional equation $P_\tau f = f$ is infeasible in all except the most trivial cases.

In [17, p. 75], Ulam conjectured that it was possible to construct finite-dimensional operators which approximate P_τ and whose fixed points approximate the fixed point of P_τ. In [13] this conjecture was proved for a class of one-dimensional piecewise expanding transformations.

The aim of this paper is to prove a version of Ulam's conjecture in a higher-dimensional setting. The main difficulty in extending the method of [13] is due to the definition of bounded variation in n dimensions which is complicated and does not possess the same intuitive properties as one-dimensional bounded variation [7]. We shall restrict our attention to a special class of higher-dimensional transformations which we shall refer to

* The research of the first author was supported by NSERC and FCAR grants.

0021-9045/91 $3.00

as Jablonski transformations. Such transformations are defined on rectangular partitions of I^n and on each element of such a partition each component of τ depends only on one variable. In spite of these restrictions, the Jablonski transformations are nontrivial extensions of the one-dimensional transformations. The Jablonski transformations are L^1 dense in the class of all piecewise expanding transformations on I^n [15]. Recently, these transformations have found an interesting application to cellular automata [6], where they are used to model the dynamics on the space of configurations.

In Section 2 we introduce the Tonelli definition of bounded variation for higher-dimensional functions [9] and state the existence theorem of Jablonski [9]. In Section 3, we obtain a generalization of the main result of [13] to Jablonski transformations in n dimensions. Unlike the strong convergence in one dimension, our result provides a weak approximation to the invariant functions. In Section 4, we discuss uniqueness of absolutely continuous invariant measures for Jablonski transformations and in Section 5 we present examples.

2. Jablonski Transformations

Let $I^n = [0, 1]^n$ and let m_j denote Lebesgue measure on I^j. For $j = n$, let $m = m_n$. We let L^1 denote the space of all Lebesgue integrable functions on I^n. The transformation $\tau: I^n \to I^n$ is written as

$$\tau(x_1, ..., x_n) = (\varphi_1(x_1, ..., x_n), ..., \varphi_n(x_1, ..., x_n)),$$

where for any $i = 1, ..., n$, $\varphi_i(x_1, ..., x_n)$ is a function from I^n into $[0, 1]$.

We say that a measurable transformation $\tau: I^n \to I^n$ is nonsingular if $m(A) = 0$ implies $m(\tau^{-1}(A)) = 0$. For nonsingular $\tau: I^n \to I^n$, we define the Frobenius–Perron operator $P_\tau : L^1 \to L^1$ by the formula

$$\int_A P_\tau f \, dx = \int_{\tau^{-1}(A)} f \, dx,$$

where $A \subseteq I^n$ is measurable. It follows that for $x = (x_1, ..., x_n)$,

$$P_\tau f(x) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\tau^{-1}(\prod_{i=1}^n [0, x_i])} f(y) \, dy.$$

It is well known that the operator P_τ is linear and satisfies the following conditions: P_τ is positive; P_τ preserves integrals; $P_{\tau^k} = P_\tau^k$, where τ^k denotes the nth iterate of τ and $P_\tau f = f$ if and only if the measure $d\mu = f \, dm$ is invariant under τ, i.e., $\mu(\tau^{-1}(A)) = \mu(A)$ for any measurable subset A of I^n.

Let $\beta = \{D_1, ..., D_p\}$ be a partition of I^n such that $p < \infty$, i.e.,

$$\bigcup_{j=1}^{p} D_j = I^n, \qquad D_j \cap D_k = \emptyset \text{ for } j \neq k.$$

A partition β of I^n is called rectangular if for any $i \leqslant j \leqslant p$, D_j is an n-dimensional rectangle.

DEFINITION 1. A transformation $\tau: I^n \to I^n$ is called a *Jablonski transformation* if it is defined on a rectangular partition and is given by the formula

$$\tau(x_1, ..., x_n) = (\varphi_{1j}(x_1), ..., \varphi_{nj}(x_n)),$$

where $(x_i, ..., x_n) \in D_j$, $1 \leqslant j \leqslant p$, $D_j = \prod_{i=1}^{n} [a_{ij}, b_{ij})$, and $\varphi_{ij} = [a_{ij}, b_{ij}] \to [0, 1]$. If $b_{ij} = 1$ for some i, then $[a_{ij}, b_{ij})$ means $[a_{ij}, b_{ij}]$.

Denote by $\prod_{i=1}^{n} A_i$ the Cartesian product of the sets A_i and by P_i the projection of R^n onto R^{n-1} given by

$$P_i(x_1, ..., x_n) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

Let $g: A \to R$ be a function on the n-dimensional interval $A = \prod_{i=1}^{n} [a_i, b_i]$. Fixing i, we define a function $\bigvee_i^A g$ of the $n-1$ variables $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ by the formula

$$\bigvee_i^A g = \bigvee_i g = \sup \Bigg\{ \sum_{k=1}^{r} | g(x_1, ..., x_i^k, ..., x_n) - g(x_1, ..., x_i^{k-1}, ..., x_n)| :$$
$$a_i = x_i^0 < x_i^1 < \cdots < x_i^r = b_i, r \in N \Bigg\}.$$

For $f: A \to R$, where $A = \prod_{i=1}^{n} [a_i, b_i]$, let

$$\mathbb{V}_i^A f = \inf \left\{ \int_{P_i(A)} \bigvee_i g \, dm_{n-1} : g = f \text{ almost everywhere, } \bigvee_i g \text{ measurable} \right\}$$

and let $\mathbb{V}^A f = \sup_{1 \leqslant i \leqslant n} \mathbb{V}_i^A f$. If $\mathbb{V}^A f < \infty$, then we say f is a bounded variation function on A and its total variation is $\mathbb{V}^A f$.

THEOREM 1 [9]. *Let the Jablonski transformation $\tau: I^n \to I^n$ on the partition $\{D_j\}_{j=1}^{p}$ be given by*

$$\tau(x_1, ..., x_n) = (\varphi_{1j}(x_1), ..., \varphi_{nj}(x_n)), (x_1, ..., x_n) \in D_j,$$

where $D_j = [a_{ij}, b_{ij})$ *if* $b_{ij} < 1$ *and* $D_{ij} = [a_{ij}, b_{ij}]$ *if* $b_{ij} = 1$, $\varphi_{ij}: [a_{ij}, b_{ij}] \to [0, 1]$ *are* C^2 *functions, and*

$$\inf_{i,j} \{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \} > 1.$$

Then for any $f \in L^1$ *the sequence* $\{(1/l) \sum_{k=0}^{l-1} P_\tau^k f\}$ *is convergent in norm to a function* $f^* \in L^1$ *as* $l \to \infty$. *The limit function has the following properties:*

(1) $f \geqslant 0$ *implies* $f^* \geqslant 0$; (2) $\int_{I^n} f^* \, dm = \int_{I^n} f \, dm$;

(3) $P_\tau f^* = f^*$ *and consequently the measure* $d\mu^* = f^* \, dm$ *is invariant under* τ (f^* *is called an invariant density*); (4) *the function* f^* *is of bounded variation. Moreover, there exists a constant* C *independent of the choice of initial* f *such that the variation of the limiting* f^* *satisfies the inequality*

$$\bigvee f^* \leqslant C \, \| f \|.$$

3. Approximating the Invariant Densities

Let $\tau: I^n \to I^n$ be a Jablonski transformation and for any positive integer l, let I^n be divided into l^n subsets of equal measure $I_1, I_2, ..., I_{l^n}$ with

$$I_k = \left[\frac{r_1}{l}, \frac{r_1+1}{l}\right) \times \left[\frac{r_2}{l}, \frac{r_2+1}{l}\right) \times \cdots \times \left[\frac{r_n}{l}, \frac{r_n+1}{l}\right)$$

for some $r_1, r_2, ..., r_n = 0, 1, ..., l-1$ and $m(I_k) = 1/l^n$, $k = 1, 2, ..., l^n$.

Let P_{st} be the fraction of I_s which is mapped into I_t by τ, i.e.,

$$P_{st} = m(I_s \cap \tau^{-1}(I_t))/m(I_s).$$

Let Δ_l be the l^n-dimensional linear subspace of L^1 which is the finite-dimensional space generated by $\{\chi_k\}_{k=1}^{l^n}$, where χ_k denotes the characteristic function of I_k, i.e., $f \in \Delta_l$ if and only if $f = \sum_{k=1}^{l^n} a_k \chi_k$ for some constants $a_1, a_2, ..., a_{l^n}$.

Define a linear operator $P_l = P_l(\tau): \Delta_l \to \Delta_l$ by

$$P_l(\tau)\, \chi_k = \sum_{t=1}^{l^n} P_{kt} \chi_t.$$

Lemmas 1–5 are straightforward n-dimensional extensions of results in [13].

Lemma 1. *Let* $\Delta_l^1 = \{\sum_{k=1}^{l^n}) a_k \chi_k : a_k \geqslant 0$ *and* $\sum_{k=1}^{l^n} a_k = 1\}$. *Then* P_l *maps* Δ_l^1 *to a subset of* Δ_l^1.

DEFINITION 2. For $f \in L^1$ and for any positive integer l we define $Q_l: L^1 \to \Delta_l$ by $Q_l f = \sum_{k=1}^{l^n} C_k \chi_k$, where $m(I_k) = 1/l^n$ and

$$C_k = \frac{1}{m(I_k)} \int_{I_k} f(x)\, dx = l^n \int_{I_k} f(x)\, dx.$$

LEMMA 2. *If $f \in L^1$ then the sequence $Q_l f$ converges in L^1 to f as $l \to \infty$.*

LEMMA 3. *If $f \in \Delta_l$, then $P_l f = Q_l P_\tau f$.*

LEMMA 4. *If $f \in \Delta_l$ then the sequence $\{P_l f\}$ converges to $P_\tau f$ in L^1 as $l \to \infty$.*

LEMMA 5. *For any integer l there exists $f_l \in \Delta_l$ such that $P_l f_l = f_l$ and $\|f_l\| = 1$; i.e., P_l has a fixed point of norm 1.*

In the course of proving Thorem 1, the following result is established in [9].

THEOREM 2. *Let τ be a Jablonski transformation, where*

$$\tau(x) = (\varphi_{1l}(x_1), ..., \varphi_{nl}(x_n)),\ x \in D_l.$$

If $\lambda = \inf_{i,l}\{\inf_{[a_{il}, b_{il}]} |\varphi'_{il}|\} > 2$, then for any $f \in L^1$,

$$\bigvee^{I^n} P_\tau f \leqslant K_\tau \|f\| + \alpha \bigvee^{I^n} f,$$

where K_τ is a constant depending on τ and $\alpha = 2\lambda^{-1} < 1$.

We require two more lemmas before we can prove our approximation result.

LEMMA 6. *If $f \in L^1$, then $\bigvee^{I^n} Q_l f \leqslant \bigvee^{I^n} f$.*

Proof. Let $I_k = \prod_{i=1}^n [(r_i/l), (r_i + 1/l)) = \prod_{i=1}^n J_{r_i}$ for some $r_i = 0, 1, ..., l-1$, $k = 1, 2, ..., l^n$ and $m(I_k) = \prod_{i=1}^n m(J_{r_i})$. Let

$$Q_{l_i} f(x) = \sum_{r_i=0}^{l-1} \left(\frac{1}{m(J_{r_i})} \int_{J_{r_i}} f(x)\, dx_i\right) \chi_{J_{r_i}}(x_i).$$

Then

$$Q_l f(x) = Q_{l_1} Q_{l_2} \cdots Q_{l_n} f(x) = \left(\prod_{i=1}^{n} Q_{l_i}\right) f(x).$$

By Lemma 2.6 of [13], we have

$$\bigvee_i^{I^n} Q_I f = \bigvee_i^{I^n} \left(\prod_{j=1}^{n} Q_{I_j} \right) f = \bigvee_i^{I^n} Q_{I_i} \left(\prod_{j=1, j \neq i}^{n} Q_{I_j} \right) f \leqslant \bigvee_i^{I^n} \left(\prod_{j=1, j \neq i}^{n} Q_{I_j} \right) f.$$

We now show that

$$\int_{I^{n-1}} \bigvee_i^{I^n} \left(\prod_{j=1, j \neq i}^{n} Q_I \right) f \left(\prod_{j=1, j \neq i}^{n} dx_j \right) \leqslant \int_{I^{n-1}} \bigvee_i^{I^n} f \left(\prod_{j=1, j \neq i}^{n} dx_j \right). \quad (1)$$

To prove (1), consider, for any $0 = x_i^0 < x_i^1 < \cdots < x_i^{r-1} < x_i^r = 1$,

$$\sum_{k=1}^{r} \left| \prod_{j=1, j \neq i}^{n} Q_{I_j} f(x_1, ..., x_i^{k-1}, ..., x_n) - \prod_{j=1, j \neq i}^{n} Q_{I_j} f(x_1, ..., x_i^k, ..., x_n) \right|$$

$$= \sum_{k=1}^{r} \left| \prod_{j=1, j \neq i}^{n} \sum_{r_j=0}^{l-1} \frac{1}{\prod_{j=1, j \neq i}^{n} m(J_{r_j})} \right.$$

$$\times \int_{\prod_{j=1, j \neq i}^{n} J_{r_j}} (f(x_1, .., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^k, ..., x_n))$$

$$\left. \left(\prod_{j=1, j \neq i}^{n} dx_j \right) \left(\prod_{j=1, j \neq i}^{n} \chi_{J_{r_j}}(x_j) \right) \right|$$

$$\leqslant \sum_{k=1}^{r} \left(\prod_{j=1, j \neq i}^{n} \sum_{r_j=0}^{l-1} \frac{1}{\prod_{j=1, j \neq 1}^{n} m(J_{r_j})} \right.$$

$$\times \int_{\prod_{j=1, j \neq i}^{n} J_{r_j}} |f(x_1, ..., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^k, ..., x_n)|$$

$$\left. \times \left(\prod_{j=1, j \neq i}^{n} dx_j \right) \left(\prod_{j=1, j \neq i}^{n} \chi_{J_{r_j}}(x_j) \right) \right).$$

Now

$$\int_{I^{n-1}} \sum_{k=1}^{r} \left| \prod_{j=1, j \neq i}^{n} Q_{I_j} f(x_1, ..., x_i^{k-1}, ..., x_n) \right.$$

$$\left. - \prod_{j=1, j \neq i}^{n} Q_{I_j} f(x_1, ..., x_1^k, ..., x_n) \right| \left(\prod_{j=1, j \neq i}^{n} dx_j \right)$$

$$
\begin{aligned}
&\leqslant \int_{I^{n-1}} \sum_{k=1}^{r} \left(\prod_{j=1, j\neq i}^{n} \sum_{r_j=0}^{l-1} \left(\prod_{j=1, j\neq i}^{n} m(J_{r_j}) \right)^{-1} \right. \\
&\quad \times \int_{\prod_{j=1, j\neq i}^{n} J_{r_j}} |f(\cdots x_i^{k-1} \cdots) - f(\cdots x_i^{k} \cdots)| \\
&\quad \left. \times \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \left(\prod_{j=1, j\neq i}^{n} \chi_{J_{r_j}}(x_j) \right) \right) \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \\
&= \sum_{k=1}^{r} \left(\prod_{j=1, j\neq i}^{n} \sum_{r_j=0}^{l-1} \int_{\prod_{j=1, j\neq i}^{n} J_{r_j}} |f(x_1, ..., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^{k}, ..., x_n)| \right. \\
&\quad \left. \times \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \frac{1}{\prod_{j=1, j\neq i}^{n} m(J_{r_j})} \int_{I^{n-1}} \prod_{j=1, j\neq i}^{n} \chi_{J_{r_j}}(x_j) \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \right) \\
&= \sum_{k=1}^{r} \int_{I^{n-1}} |f(x_1, ..., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^{k}, ..., x_n)| \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \\
&= \int_{I^{n-1}} \sum_{k=1}^{r} |f(x_1, ..., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^{k}, ..., x_n)| \left(\prod_{j=1, j\neq i}^{n} dx_j \right).
\end{aligned}
$$

Hence,

$$
\int_{I^{n-1}} \bigvee_i^{I^n} \left(\prod_{j=1, j\neq i}^{n} Q_{l_j} \right) f \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \leqslant \int_{I^{n-1}} \bigvee_i^{I^n} f \left(\prod_{j=1, j\neq i}^{n} dx_j \right).
$$

Now,

$$
\begin{aligned}
\mathbb{V}_i^{I^n} Q_l f &= \inf \left\{ \int_{I^{n-1}} \bigvee_i^{I^n} h \left(\prod_{j=1, j\neq i}^{n} dx_j \right), h = Q_l f \text{ a.e.}, \bigvee_i^{I^n} h \text{ measurable} \right\} \\
&\leqslant \inf \left\{ \int_{I^{n-1}} \bigvee_i^{I^n} Q_l g \left(\prod_{j=1, j\neq i}^{n} dx_j \right), g = f \text{ a.e.}, \bigvee_i^{I^n} Q_l g \text{ measurable} \right\} \\
&\leqslant \inf \left\{ \int_{I^{n-1}} \bigvee_i^{I^n} \left(\prod_{j=1, j\neq i}^{n} Q_{l_j} \right) g \left(\prod_{j=1, j\neq i}^{n} dx_j \right), \right. \\
&\qquad\qquad \left. g = f \text{ a.e.}, \bigvee_i^{I^n} g \text{ measurable} \right\} \\
&\leqslant \inf \left\{ \int_{I^{n-1}} \bigvee_i^{I^n} g \left(\prod_{j=1, j\neq i}^{n} dx_j \right), g = f \text{ a.e.}, \bigvee_i^{I^n} g \text{ measurable} \right\} = \mathbb{V}_i^{I^n} f.
\end{aligned}
$$

Therefore $\mathbb{V}^{I^n} Q_l f = \max_i \mathbb{V}_i^{I^n} Q_l f \leqslant \max_i \mathbb{V}_i^{I^n} f = \mathbb{V}^{I^n} f$. In the foregoing argument we have used the fact that for any positive integer l and $f, g \in L^1$, $f = g$ a.e. implies $Q_l f = Q_l g$ a.e. and $\bigvee_i^{I^n} g$ measurable implies $\bigvee_i^{I^n} Q_l g$ measurable. ∎

LEMMA 7. *Let τ be a Jablonski transformation*

$$\tau(x) = (\varphi_{1j}(x_j), ..., \varphi_{nj}(x_n)), \quad x \in D_j$$

and $f_l \in \Delta_l$ be any fixed point of $P_l(\tau)$ with $\| f_l \| = 1$. If

$$\lambda = \inf_{i,j} \{ \inf_{[a_{ij}, b_{ij}]} | \varphi'_{ij} | \} > 2,$$

then the sequence $\{ \bigvee^{I^n} f_l \}_{l=1}^{\infty}$ is bounded.

Proof. By Lemma 3, $f_l = P_l f_l = Q_l P_\tau f_l$ for all l. Hence by Theorem 2 and Lemma 6.

$$\bigvee^{I^n} f_l = \bigvee^{I^n} Q_l P_\tau f_l \leqslant \bigvee^{I^n} P_\tau f_l \leqslant K_\tau \| f_l \| + \alpha \bigvee^{I^n} f_l = K_\tau + \alpha \bigvee^{I^n} f_l,$$

where $K_\tau > 0$ and $0 < \alpha < 1$. Since $\bigvee^{I^n} f_l < \infty$, we have $\bigvee^{I^n} f_l \leqslant K_\tau/(1-\alpha)$. ∎

The following self-adjoint property of Q_l which was not needed in [13] plays a vital role in the sequel.

LEMMA 8. *For any $f \in L^1$, $l = 1, 2, ...$, and measurable subset A of I^n*

$$\int_{I^n} \chi_A Q_l f \, dx = \int_{I^n} f Q_l \chi_A \, dx.$$

Proof.

$$\begin{aligned}
\int_{I^n} \chi_A(x) Q_l f(x) \, dx &= \int_{I^n} \chi_A(x) \left(\sum_{k=1}^{l^n} \frac{1}{m(I_k)} \int_{I_k} f(y) \, dy \, \chi_k(x) \right) dx \\
&= \sum_{k=1}^{l^n} \frac{1}{m(I_k)} \int_{I_k} f(y) \, dy \int_{I^n} \chi_A(x) \chi_k(x) \, dx \\
&= \sum_{k=1}^{l^n} \frac{m(A \cap I_k)}{m(I_k)} \int_{I_k} f(y) \, dy = \sum_{k=1}^{l^n} \frac{m(A \cap I_k)}{m(I_k)} \int_{I_k} f(x) \, dx \\
&= \int_{I^n} f(x) \left(\sum_{k=1}^{l^n} \frac{1}{m(I_k)} \int_{I_k} \chi_A(y) \, dy \, \chi_k(x) \right) dx \\
&= \int_{I^n} f(x) Q_l \chi_A(x) \, dx. \quad ∎
\end{aligned}$$

THEOREM 3. *Let τ be a nonsingular Jablonski transformation with partition $\{D_1, ..., D_p\}$ and $\lambda = \inf_{i,j} \{ \inf_{[a_{ij}, b_{ij}]} | \varphi'_{ij} | \} > 2$. Suppose P_τ has a unique fixed point. Then for any positive integer l, $P_l(\tau)$ has a fixed point f_l in Δ_l*

with $\|f_l\| = 1$ *and the sequence* $\{f_l\}$ *converges weakly to the fixed point of* P_τ.

Proof. By Lemma 7 and Lemma 3 of [9], we know that the set $\{f_l\}_{l=1}^{\infty}$ is weakly relatively compact in L^1. Let $\{f_{l_j}\}$ be any weakly convergent subsequence of $\{f_l\}_{l=1}^{\infty}$ and let $f = \lim_{j\to\infty} f_{l_j}$ weakly. Then for any $g \in L^\infty$,

$$\left|\int_{I^n} g(f - P_\tau f)\,dx\right| \leq \left|\int_{I^n} g(f - f_{l_j})\,dx\right| + \left|\int_{I^n} g(f_{l_j} - Q_{l_j}P_\tau f_{l_j})\,dx\right|$$
$$+ \left|\int_{I^n} g(Q_{l_j}P_\tau f_{l_j} - P_\tau f)\,dx\right|.$$

The first term approaches 0 since f_{l_j} converges weakly to f as $j\to\infty$. By Lemma 3, $Q_{l_j}P_\tau f_{l_j} = P_{l_j} f_{l_j} = f_{l_j}$. The second term is identically 0.

We now consider the last term. By the weak continuity of P_τ [11, p. 43], $P_\tau f_{l_j}$ converges weakly to $P_\tau f$ as $j\to\infty$. We will prove that $Q_{l_j}P_\tau f_{l_j}$ converges weakly to $P_\tau f$ as $j\to\infty$. It is enough to show that for any measurable subset A of I^n we have

$$\lim_{j\to\infty}\int_{I^n} \chi_A Q_{l_j} h_{l_j}\,dx = \int_{I^n} \chi_A h\,dx,$$

where $h_{l_j} = P_\tau f_{l_j}$ and $h = P_\tau f$.

By Corollary IV.8.11 in [4, p. 294],

$$\int_E h_{l_j}(x)\,dx \to 0 \text{ as } m(E)\to 0$$

uniformly in j. Because $\|h_{l_j}\| = 1$ and $h_{l_j} \geq 0$, by Theorem 7.5.3 in [1, p. 296], h_{l_j} are uniformly integrable, i.e.,

$$\int_{\{|h_{l_j}|\geq K\}} |h_{l_j}|\,dx \to 0 \qquad \text{as} \quad K\to\infty$$

uniformly in j. Therefore, for any $\varepsilon > 0$, there exists $K > 0$ such that for all j

$$2\int_{\{|h_{l_j}|\geq K\}} |h_{l_j}|\,dx < \varepsilon.$$

Hence

$$\left|\int_{I^n} h_{l_j}(Q_{l_j}\chi_A - \chi_A)\,dx\right|$$
$$\leq \int_{I^n} |h_{l_j}|\,|Q_{l_j}\chi_A - \chi_A|\,dx$$

$$= \int_{\{|h_{l_j}| \geq K\}} |h_{l_j}| \, |Q_{l_j}\chi_A - \chi_A| \, dx + \int_{\{|h_{l_j}| < K\}} |h_{l_j}| \, |Q_{l_j}\chi_A - \chi_A| \, dx$$

$$\leq 2 \int_{\{|h_{l_j}| \geq K\}} |h_{l_j}| \, dx + K \int_{\{|h_{l_j}| < K\}} |Q_{l_j}\chi_A - \chi_A| \, dx$$

$$\leq 2 \int_{\{|h_{l_j}| \geq K\}} |h_{l_j}| \, dx + K \int_{I^n} |Q_{l_j}\chi_A - \chi_A| \, dx.$$

The first term is less than ε and the second term approaches 0 as $j \to \infty$ by Lemma 2. Thus

$$\lim_{j \to \infty} \int_{I^n} h_{l_j}(Q_{l_j}\chi_A - \chi_A) \, dx = 0.$$

By Lemma 8,

$$\lim_{j \to \infty} \int_{I^n} \chi_A Q_{l_j} h_{l_j} \, dx = \lim_{j \to \infty} \int_{I^n} h_{l_j} Q_{l_j} \chi_A \, dx$$

$$= \lim_{j \to \infty} \int_{I^n} h_{l_j}(Q_{l_j}\chi_A - \chi_A) \, dx + \lim_{j \to \infty} \int_{I^n} h_{l_j} \chi_A \, dx$$

$$= \int_{I^n} h \chi_A \, dx.$$

This means the last term approaches 0.

We have, therefore, established that for any $g \in L^\infty$,

$$\int_{I^n} g(x) \, (f(x) - P_\tau f(x)) \, dx = 0.$$

This means $P_\tau f(x) = f(x)$ almost everywhere. Therefore any weakly convergent subsequence of $\{f_l\}$ converges weakly to a unique fixed point of P_τ. Hence $f_l \to f$ weakly as $l \to \infty$. ∎

COROLLARY 1. *If the fixed point of P_τ is not unique in Theorem 3, then any weak limit point of $\{f_l\}_{l=1}^\infty$ is a fixed point of P_τ.*

THEOREM 4. *Let τ be a nonsingular Jablonski transformation with $\lambda = \inf_{i,j} \{\inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}|\} > 1$. Suppose P_τ has a unique fixed point. Let k be an integer such that $\lambda^k > 2$. Let $\phi = \tau^k$ and f_l be a fixed point of $P_l(\phi)$. Let*

$$g_l = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f_l.$$

Then $\{g_l\}$ converges weakly to the fixed point of P_τ.

Proof. Since P_τ is a weakly continuous operator [11, p. 43], Theorem 3 implies that $g_l \to g = (1/k) \sum_{j=0}^{k-1} P_{\tau^j} f$ weakly as $l \to \infty$. Therefore

$$P_\tau g = \frac{1}{k} \sum_{j=1}^{k} P_{\tau^j} f = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f = g,$$

where f is the fixed point of $P_\phi = P_{\tau^k}$, i.e., $P_{\tau^k} f = f$. ▮

COROLLARY 2. *If the fixed point of P_τ is not unique in Theorem 4, then any weak limit point f of $\{f_l\}_{l=1}^{\infty}$ is a fixed point of P_ϕ and $g = (1/k) \sum_{j=0}^{k-1} P_{\tau^j} f$ is a fixed point of P_τ. If $f_{l_i} \to f$ weakly as $i \to \infty$ then $g_{l_i} = (1/k) \sum_{j=0}^{k-1} P_{\tau^j} f_{l_i} \to g$ weakly as $i \to \infty$.*

4. UNIQUENESS OF INVARIANT DENSITIES

Let $\tau: I^n \to I^n$ be a Jablonski transformation. Without loss of generality we shall assume there exist

$$0 = a_{i,0} < a_{i,1} < \cdots < a_{i,r_i} = 1, \qquad i = 1, 2, ..., n$$

for some positive integers $r_1, r_2, ..., r_n$ such that the partition β is composed of sets $D_{s_1, ..., s_n} = \prod_{i=1}^{n} D_{s_i}$, where $D_{s_i} = [a_{i,s_i-1}, a_{i,s_i})$, $s_i = 1, 2, ..., r_i - 1$, $D_{r_i} = [a_{i,r_i-1}, a_{i,r_i}]$, and τ is given by the formula

$$\tau(x) = (\varphi_{1,s_1,...,s_n}(x_1), ..., \varphi_{n,s_1,...,s_n}(x_n)), \; x \in D_{s_1,...,s_n},$$

where $\varphi_{i,s_1,...,s_n}: \bar{D}_{s_i} \to [0, 1]$ are C^2 functions.

DEFINITION 3. We say that the partition β has the *communication property* under the transformation $\tau: I^n \to I^n$ if for any elements $D'_{s_1,...,s_n}$ and $D''_{s_1,...,s_n}$ of β there exist integers u and v such that $D'_{s_1,...,s_n} \subset \tau^u(D''_{s_1,...,s_n})$ and $D''_{s_1,...,s_n} \subset \tau^v(D'_{s_1,...,s_n})$.

DEFINITION 4. A Jablonski transformation $\tau: I^n \to I^n$ is in class $\mathscr{C}$ if it satisfies following conditions for the fixed partition β:

(1) $\inf |\varphi'_i| > 0$ and $\inf |(\varphi_i^w)'| > 1$ for some integer w;

(2) τ is piecewise C^2;

(3) the partition β has the communication property under τ.

We associate with each $D_{s_1,...,s_n}$ a symbol such as $\alpha, \beta, \gamma, ...$, and code the orbit by a sequence $\langle x \rangle = \alpha \beta \gamma \cdots$ if $x \in D(\alpha)$, $\tau(x) \in D(\beta)$, $\tau^2(x) \in D(\gamma)$, ..., where $D(\alpha)$ is some $D_{s_1,...,s_n}$ whose symbol is α. The following three lemmas are identical to the one-dimensional versions proved in [2].

LEMMA 9. *Let $\tau: I^n \to I^n$ be a C^2 Jablonski transformation which satisfies condition* (1) *defining class $\mathscr{C}$. Then $\langle x \rangle = \langle y \rangle$ implies $x = y$.*

LEMMA 10. *Let τ be as in Lemma 9. If $\sigma = \alpha_1 \alpha_2 \cdots$ is a sequence with the property that $\tau(D(\alpha_k)) \supset D(\alpha_{k+1})$, $k = 1, 2, \ldots$, then there exists a unique $x \in I^n$ such that $\langle x \rangle = \sigma$.*

LEMMA 11. *Let τ be the same as in Lemma 9 and let $\xi \subset \beta$ be a collection of elements satisfying the communication property: for any $D_1, D_2 \in \xi$, there exist integers u and v such that $D_1 \subset \tau^u(D_2)$ and $D_2 \subset \tau^v(D_1)$. Assume that ξ contains at least two D_i's and $V = \bigcup_{D \in \xi} D$. Then there exists an $x \in V$ such that $\{\tau^l(x)\}_{l=1}^{\infty}$ is dense in V.*

LEMMA 12. *If τ is the same as in Lemma* 11 *and satisfies condition* (3) *defining class $\mathscr{C}$, then there exists a dense orbit in all of I^n.*

THEOREM 5. *If $\tau \in \mathscr{C}$ then the absolutely continuous invariant measure under τ is unique.*

Proof. Assume there exist two such measure with densities f_1 and f_2. As in [14], it can be shown that there exist two invariant functions $f_1^* \geq 0$, $f_2^* \geq 0$, $\| f_1^* \| = \| f_2^* \| = 1$ such that $S_1 = \operatorname{spt} f_1^*$ and $S_2 = \operatorname{spt} f_2^*$ are disjoint and S_i is an union of disjoint regions, $i = 1, 2$. From [8] we know that each S_i has interior.

Now let $x \in I^n$ be a point which has a dense orbit in I^n. By Lemma 12 such a point exists. The denseness of the orbit $\{\tau^l(x)\}_{l=1}^{\infty}$ implies there exist points $u = \tau^{l_1}(x)$ and $v = \tau^{l_2}(u)$ such that $u \in \operatorname{int} S_1$ and $v \in \operatorname{int} S_2$, where int denotes interior. By the piecewise continuity of τ there exists an open ball O_1 centered at u and in S_1 such that for $u \in O_1$, $v = \tau^{l_2}(u) \in \operatorname{int} S_2$. But S_1 and S_2 are invariant sets [14], i.e., $\tau(S_i)$ a.e., $i = 1, 2$. Hence, we have a contradiction. Therefore, there exists only one absolutely continuous invariant measure under τ. ∎

THEOREM 6. *Let $\tau: I^n \to I^n$ be a Jablonski transformation with the partition $\beta = \{D_{s_1, \ldots, s_n}\}$, $s_i = 1, 2, \ldots, r_i$, $i = 1, 2, \ldots, n$ given by the formula*

$$\tau(x) = (\varphi_{1, s_1, \ldots, s_n}(x_1), \ldots, \varphi_{n, s_1, \ldots, s_n}(x_n)), \; x \in D_{s_1, \ldots, s_n}$$

such that: (1) *for any $s_1, \ldots, s_n$ and i, $\varphi_{i, s_1, \ldots, s_n}(x_i) \in C^2$ and $|\varphi'_{i, s_1, \ldots, s_n}(x_i)| \geq \lambda > 1$; and* (2) *the partition β has the communication property with respect to τ. Then P_τ has a unique fixed point f with $\| f \| = 1$.*

Proof. *τ satisfies all the conditions of Theorem 5.* ∎

5. Examples

(1) If for any element $D_{s_1, \ldots, s_n}$ of β, $\varphi_{i, s_1, \ldots, s_n}$ is a C^2 bijective map of the closed interval $\bar{D}_{s_i}$ onto $[0, 1]$, the restriction $\tau_{s_1, \ldots, s_n}$ of τ on $D_{s_1, \ldots, s_n}$ is a C^2 bijective transformation of $\bar{D}_{s_1, \ldots, s_n}$ onto I^n and

$$\lambda = \inf |\varphi'_{i, s_1, \ldots, s_n}| > 1,$$

then $\tau \in \mathscr{C}$ and by Theorem 5 the absolutely continuous invariant measure under τ is unique. Furthermore, If $\lambda > 2$ then by Theorem 3 we have a sequence of piecewise constant functions f_l with $\| f_l \| = 1$ which converges weakly to the fixed point of P_τ.

(2) We now present an example for $n = 2$, where the elements of the partition β do not map onto all of I^2. (See Fig. 1.) Let $I_1 = J_1 = [0, \frac{1}{4})$, $I_2 = J_2 = [\frac{1}{4}, \frac{1}{2})$, $I_3 = J_3 = [\frac{1}{2}, \frac{3}{4})$, $I_4 = J_4 = [\frac{3}{4}, 1]$ and $D_{kj} = I_k \times J_j$, k, $j = 1, 2, 3, 4$.
Let $h_1(x) = 2.4(x^2 + x)$, $h_2(x) = h_1(x - \frac{1}{4})$, $h_3(x) = h_1(x - \frac{1}{2})$, $h_4(x) = h_1(x - \frac{3}{4})$, $g_i(y) = h_i(y)$, $i = 1, 2, 3, 4$, $h(x) = 4x$, $g(y) = 4y$. Then define

$$\tau(x, y) = \begin{cases} (h_k(x), g_j(y)), & (x, y) \in D_{kj}, \quad D_{kj} \neq D_{11}, \\ (h(x), g(y)), & (x, y) \in D_{11}. \end{cases}$$

Since $\tau(\bar{D}_{kj}) = [0, \frac{3}{4}] \times [0, \frac{3}{4}]$, $(D_{kj} \neq D_{11})$, $\tau(\bar{D}_{11}) = I^2$. By [8] we know that P_τ has a fixed point and by Theorem 6 it is unique. Also in view of Theorem 3, $f_l \in \Delta_l$ with $\| f_l \| = 1$ and $\{f_l\}$ converges weakly to the fixed point of P_τ as $l \to \infty$.

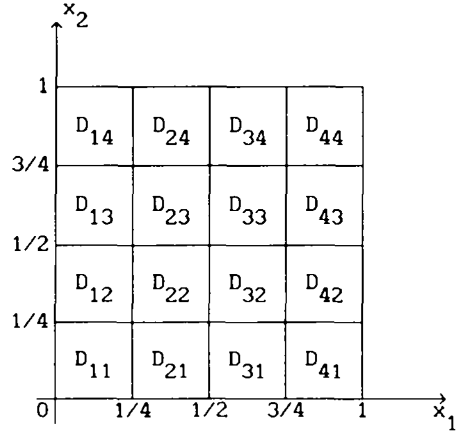

Fig. 1. The domain of a two-dimensional Jablonski transformation.

REFERENCES

1. ROBERT B. ASH, "Real Analysis and Probability," Academic Press, New York, 1972.
2. ABRAHAM BOYARSKY AND MANNY SCAROWSKY, On a class of transformations which have unique absolutely continuous invariant measures, *Trans. Amer. Math. Soc.* **255** (1979), 243–262.
3. D. CANDELORA, Misure invariante per transformazioni in piu dimensioni, *Atti Sem. Mat. Fis. Univ. Modena* **35** (1987), 33–42.
4. DUNFORD AND SCHWARTZ, "Linear Operators. Part I. General Theory," Wiley, New York, 1963.
5. P. GORA AND A. BOYARSKY, Existence of absolutely continuous invariant measures for n-dimensional transformations, *Israel J. Math.*, in press.
6. P. GORA AND A. BOYARSKY, Higher dimensional point transformations and asymptotic measures for cellular automata, *Comput. Math. Appl.* **19** (1990), 13–31.
7. P. GORA AND A. BOYARSKY, On functions of bounded variation in higher dimensions, *Amer. Math. Monthly*, in press.
8. P. GORA AND A. BOYARSKY, On the number of absolutely continuous measures invariant under higher dimensional transformations, preprint, 1989.
9. M. JABLONSKI, On invariant measures for piecewise C^2-transformation of the n-dimensional cube, *Ann. Polon. Math.* **43** (1983), 185–195.
10. GERHARD KELLER, Probabilités–ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan, *C. R. Acad. Sci. Paris, Sér. A* **289** (1979), 625–627.
11. A. LASOTA AND M. MACKEY, "Probabilistic Properties of Deterministic Systems," Cambridge Univ. Press, 1985.
12. A. LASOTA AND J. A. YORKE, On the existence of invariant measure for piecewise monotonic transformation, *Trans. Amer. Math. Soc.* **186** (1973), 481–488.
13. TIEN-YIEN LI, Finite approximation for Frobenius–Perron operator: A solution to Ulam's conjecture, *J. Approx. Theory* **17** (1976), 177–186.
14. TIEN-YIEN LI AND J. A. YORKE, Ergodic transformations from an interval into itself, *Trans. Amer. Math. Soc.* **235** (1978), 183–192.
15. Y. S. LOU, "Existence and Approximation of Absolutely Continuous Invariant Measures for Higher Dimensional Transformations," Doctoral Dissertation, Concordia University, 1991.
16. F. SCHWEIGER, Invariant measures and ergodic properties of number theoretical endomorphisms, Banach Center Publications, to appear.
17. S. ULAM, "Problems in Mathematics," Interscience, New York, 1960.